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Generalized Heisenberg ferromagnet models via Hermitian symmetric spaces

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Abstract

We study the generalized Heisenberg ferromagnet model using a duality with the Hermitian symmetric space nonlinear Schrödinger equation. We explain the mathematical structure of the Hermiticity of the space in connection with the soliton surface theory and obtain various spin configurations in N-dimension using this correspondence.

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1. Introduction

Recently, there has been renewed interest to the theory of 'soliton surfaces' which establishes a general relation between integrable systems and integrable geometries [1, 2]. It provides an interesting connection between the nonlinear dynamics of certain physical systems described in terms of moving curves and surfaces and the corresponding nonlinear partial differential equations [3]. A classic example is the equation of spin dynamics of the Heisenberg ferromagnet (HF) model, which was shown to be equivalent to the nonlinear Schrödinger (NS) equation. Since Lakshmanan *et al* [4] first showed that the continuum dynamics of the HF model is gauge equivalent to the NS equation, there appear many studies on this topic including the result that describes a classical inhomogeneous Heisenberg chain in terms of an integro-differential NS equation [5]. On the other hand, Sym [6] introduced the Sym (or Sym–Tafel) formula, which associates the dynamical variables of physical systems with the linear problems of corresponding integrable equations.

Later on, Orfanidis [7] introduced an integrable $SU(n)$-invariant Heisenberg spin chain which is associated with the $U(n-1)$-invariant NS equation. But the intrinsic mathematical structure consequent upon the extension to higher dimension was not understood until Fordy and Kulish [8] introduced the NS equation based on the Hermitian symmetric spaces and its corresponding generalized HF model. Its Lax pair formalism as well as conservation laws are also developed, consecutively [9]. Even though Fordy and Kulish succeeded in introducing the Hermitian symmetric spaces, their result, including their successor, was essentially a gauge equivalence between the spin variables of the HF model and the variables of the Hermitian

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symmetric space NS equation. So it was difficult to calculate explicitly spin motions of the HF model, and the full understanding of its intrinsic mathematical structure was still missing.

In this paper, we will introduce a Sym-type formula that describes the correspondence between the generalized HF model and the Hermitian symmetric space NS equation. We also clarify the mathematical structure of the higher-dimensional formalism, showing that the complex structure of the Hermitian symmetric space is indispensable for the Sym-type formalism. Unlike the previous studies where the Hermitian symmetric spaces are treated in an abstract way, we introduce M matrices, which represent concretely an important subset of the Hermitian symmetric spaces. The 1-solitonic spin motion of the generalized HF models is explicitly calculated using the decomposition through M matrices. Spin motions corresponding to the plane wave solutions of the NS equation are also obtained. Further more, the NS equation was known to be related with the motion of a vortex filament in a fluid [10]. Using our Sym-type formula, this duality can be easily extended to the case of the Hermitian symmetric spaces, as well.

2. Hermitian symmetric space NS equation

2.1. Hermitian symmetric space

The complex structure of the Hermitian symmetric space [11] is essential for the soliton surface theory of spin motions. Let $\boldsymbol{g}$ and $\boldsymbol{k}$ be associated Lie algebras for G/K whose orthogonal decomposition, $\boldsymbol{g} = \boldsymbol{k} \oplus \boldsymbol{m}$, satisfies the commutation relations,

$$[\boldsymbol{k}, \boldsymbol{k}] \subset \boldsymbol{k} \qquad [\boldsymbol{k}, \boldsymbol{m}] \subset \boldsymbol{m} \qquad [\boldsymbol{m}, \ \boldsymbol{m}] \subset \boldsymbol{k}. \tag{1}$$

The Hermiticity of G/K implies that there exists an element T in the Cartan subalgebra of $\boldsymbol{k}$ whose adjoint action defines a complex structure such that $[T, [T, m]] = -m$ for $m \in \boldsymbol{m}$. The subalgebra $\boldsymbol{k}$ is characterized by the property that it commutes with T, i.e., $[T, k] = 0$ for $k \in \boldsymbol{k}$.

The complete classification of Hermitian symmetric spaces is known [11] in terms of four series and two exceptional cases; AIII $= SU(m+n)/(SU(m) \times SU(n) \times U(1))$, CI $= Sp(n)/U(n)$, DIII $= SO(2n)/U(n)$, BDI $= SO(m+2)/(SO(m) \times SO(2))$, EIII $= E_6/(SO(10) \times SO(2))$, EVII $= E_7/(E_6 \times SO(2))$.

Especially for AIII, CI and DIII series, $(m+n) \times (m+n)$ matrices $E \in \boldsymbol{m}$ and T can be rewritten in block forms using M matrices,

$$E = \begin{pmatrix} 0 & M \\ -\sigma M^\dagger & 0 \end{pmatrix} \qquad T = \frac{\mathrm{i}}{2} \begin{pmatrix} I_{m\times m} & 0 \\ 0 & -I_{n\times n} \end{pmatrix} \tag{2}$$

where $\dagger$ denotes Hermitian conjugate and σ is a real $n \times n$ matrix with the property $\sigma^2 = 1, \sigma^\dagger = \sigma$. The matrix M is a complex $m \times n$ matrix for AIII, a complex symmetric $n \times n$ matrix $(m = n)$ for CI, and a complex anti-symmetric $n \times n$ matrix $(m = n)$ for DIII. Spaces associated with $\sigma = I_{n\times n}$ are named as type-I, while others are named as type-II. Note that E has the generalized Hermitian property, $E^\dagger = -\Sigma E \Sigma$ where $\Sigma = \mathrm{diag}(I_{m\times m}, \sigma)$.

The Hermitian symmetric space NS equation associated with the symmetric space G/K takes the form [12];

$$\bar{\partial} E = -\partial^2 \tilde{E} + \tfrac{1}{2}[E, \ [E, \ \tilde{E}]] \tag{3}$$

where $\partial \equiv \partial/\partial z$, $\bar{\partial} \equiv \partial/\partial \bar{z}$ and $\tilde{E} \equiv [T, \ E] \in \boldsymbol{m}$. Here $E(z, \bar{z})$ denotes the field variable of the Hermitian symmetric space NS equation. For spaces AIII, CI and DIII, it can be rewritten using the matrix M as

$$\bar{\partial} M = -\mathrm{i}\partial^2 M - 2\mathrm{i} M \sigma M^\dagger M. \tag{4}$$

Here we give some typical examples.

Example 1. AIII type-I ($m = 1, n = 2$).
This model, named as $CP(2)$ space, corresponds to taking $G = SU(3), K = SU(2) \times U(1), \sigma = I_{2\times 2}$. The subalgebra $\boldsymbol{m}$ is comprised of four elements, $\lambda_1, \lambda_2, \lambda_4, \lambda_5$ from the eight Gell-Mann generators λ_i of the $\boldsymbol{su(3)}$ algebra.

E, M and T matrices are

$$E = \begin{pmatrix} 0 & \psi_1 & \psi_2 \\ -\psi_1^* & 0 & 0 \\ -\psi_2^* & 0 & 0 \end{pmatrix} \qquad M = (\,\psi_1 \quad \psi_2\,) \qquad T = \frac{\mathrm{i}}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \frac{1}{6} I_{3\times 3} \tag{5}$$

where the tracelessness of T requires a term $\frac{1}{6}I$ compared to the form in equation (2). But this does not introduce any noticeable difference for our immersion formulation. Then equation (4) becomes the vector NS equation;

$$\bar{\partial}\psi_k = -\mathrm{i}\partial^2\psi_k - 2\mathrm{i}(|\psi_1|^2 + |\psi_2|^2)\psi_k \qquad k = 1, 2. \tag{6}$$

Example 2. AIII type-II ($m = n = 1$).
This model corresponds to taking $G = SU(1, 1), K = U(1), \sigma = -1$ with the $\boldsymbol{m}$ subalgebra comprised of $T_1 \equiv \sigma_1, T_2 \equiv \sigma_2$, where σ_i are the Pauli matrices. The $\boldsymbol{k}$ subalgebra has one element $T_3 \equiv \mathrm{i}\sigma_3 = 2T$ and the E matrix is

$$E = \begin{pmatrix} 0 & \psi \\ \psi^* & 0 \end{pmatrix}. \tag{7}$$

Equation (4) now becomes the NS equation with normal dispersion,

$$\bar{\partial}\psi = -\mathrm{i}\partial^2\psi + 2\mathrm{i}|\psi|^2\psi. \tag{8}$$

2.2. *Lax pair*

The Lax pair of Hermitian symmetric space NS equations is given by

$$\begin{aligned} 0 &= L_z\Psi \equiv \big[\partial + E + \lambda T\big]\Psi \\ 0 &= L_{\bar{z}}\Psi \equiv \big[\bar{\partial} + \tfrac{1}{2}[E,\ \tilde{E}] - \partial\tilde{E} - \lambda E - \lambda^2 T\big]\Psi. \end{aligned} \tag{9}$$

One can check that the Hermitian symmetric space NS equation in equation (3) is equivalent to the compatibility condition of over-determined linear equations; $[L_z,\ L_{\bar{z}}] = 0$ which holds for all values of λ. The linear equation in equation (9) can be rewritten in terms of the M matrix,

$$\begin{aligned} 0 &= \left[\partial + \frac{\mathrm{i}}{2}\lambda\right]\Psi_1 + M\Psi_2 \\ 0 &= \left[\partial - \frac{\mathrm{i}}{2}\right]\Psi_2 - \sigma M^\dagger\Psi_1 \\ 0 &= \left[\bar{\partial} + \mathrm{i}M\sigma M^\dagger - \frac{\mathrm{i}}{2}\lambda^2\right]\Psi_1 - \big[\mathrm{i}\partial M + \lambda M\big]\Psi_2 \\ 0 &= \left[\bar{\partial} - \mathrm{i}\sigma M^\dagger M + \frac{\mathrm{i}}{2}\lambda^2\right]\Psi_2 + \big[\lambda\sigma M^\dagger - \mathrm{i}\sigma\partial M^\dagger\big]\Psi_1 \end{aligned} \tag{10}$$

where Ψ_1 and Ψ_2 are $m \times (m+n)$ and $n \times (m+n)$-dimensional matrices, respectively.

2.3. *Darboux transformation*

The Darboux transformation (DT) is used to obtain a set of new solutions $E[N]$, $\Psi[N]$ from a known set of E, Ψ in equation (9) [13]. In the next section we will use the DT to obtain a new solution of the HF model from a known one. It is defined by

$$\Psi[N] = [\lambda - \lambda_1^* - (\lambda_1 - \lambda_1^*)P]\Psi \qquad E[N] = E + (\lambda_1 - \lambda_1^*)[T,\ P]. \tag{11}$$

Here the projection operator P ($P^2 = P$) is

$$P = \frac{\Phi\Phi^\dagger\Sigma}{\Phi^\dagger\Sigma\Phi} \tag{12}$$

where the *column matrix* Φ is a solution of the linear equation in equation (9) at a specific value of $\lambda = \lambda_1$ (the DT parameter).

3. Dualization

3.1. Heisenberg ferromagnet model

The HF system was first introduced to study the spin motion when they interact with the nearest neighbors only. Under the continuum limit, it becomes the classical spin system that can be described by the following equation,

$$\frac{\partial \boldsymbol{S}(x,t)}{\partial t} = \boldsymbol{S}(x,t) \times \frac{\partial^2 \boldsymbol{S}(x,t)}{\partial x^2}. \tag{13}$$

This equation was shown to be integrable and their N-soliton solutions were found by the inverse scattering method [14].

The spin equation (13), describing the spin motion in three-dimensional space, can be written down in terms of $SU(2)$ matrix variable $Q(z,\bar{z}) \equiv \frac{1}{2}\boldsymbol{S}\cdot\boldsymbol{\sigma}$ as

$$\bar{\partial}Q = -[Q,\ \partial^2 Q] \tag{14}$$

where σ denotes the Pauli matrix and $z = x, \bar{z} = t, \partial = \frac{\partial}{\partial z}, \bar{\partial} = \frac{\partial}{\partial \bar{z}}$. This equation was extended to the $SU(N)$ case in an obvious way in [7].

3.2. Immersion

We first find the duality between the HF and the NS model using the correspondence developed in the theory of 'soliton surfaces'. To do this, we introduce a Sym–Tafel type formula [15]

$$Q(z,\bar{z}) = \Psi^{-1}T\Psi \tag{15}$$

which associates with a given Hermitian symmetric space NS model an immersion Q into the Lie algebra of the linear problem in equation (9). The immersion Q is related to the generalized spin S of the HF model extended to N-dimensions of Hermitian symmetric spaces as

$$Q = \alpha \sum_{i=1}^{\dim g} S_i T_i \tag{16}$$

where α is a normalization constant.

In order to prove that the immersion Q indeed satisfy the equation of motion in equation (14), as well as the constraint Tr $Q^2 =$ constant, we first note that

$$\bar{\partial}Q = -\Psi^{-1}\bar{\partial}\Psi\Psi^{-1}T\Psi + \Psi^{-1}T\bar{\partial}\Psi = \Psi^{-1}[T,\ \bar{\partial}\Psi\Psi^{-1}]\Psi. \tag{17}$$

Using the linear equation in (9) for $\bar{\partial}\Psi$ and the following property from the Hermitian symmetric space,

$$[T,\ [E,\ \tilde{E}]] \sim [T,\ [\boldsymbol{m},\ \boldsymbol{m}]] \sim [T,\ \boldsymbol{k}] = 0 \tag{18}$$

we obtain

$$\bar{\partial}Q = \Psi^{-1}(\lambda\tilde{E} - \partial E)\Psi. \tag{19}$$

In a similar way, we find that $\partial Q = -\Psi^{-1}\tilde{E}\Psi$. Another application of ∂ on ∂Q gives

$$\partial^2 Q = \Psi^{-1}\big([\partial\Psi\Psi^{-1},\ \tilde{E}] - \partial\tilde{E}\big)\Psi = -\Psi^{-1}\big([E+\lambda T,\ \tilde{E}] + \partial\tilde{E}\big)\Psi. \tag{20}$$

Using the property in equation (18) and $[T,\ [T,\boldsymbol{m}]] = -\boldsymbol{m}$, we obtain

$$[Q,\ \partial^2 Q] = \Psi^{-1}(\lambda\tilde{E} + \partial E)\Psi. \tag{21}$$

Now it is easy to see that

$$\bar{\partial}Q = -2\lambda\partial Q - [Q,\ \partial^2 Q] \tag{22}$$

which becomes the immersion equation in (14) when we set[1] $\lambda = 0$. It means that we can obtain a solution for the spin equation from a solution of the linear equation of the NS model. The constraint $S^2 = 1$ results from $\operatorname{Tr} Q^2 = \operatorname{Tr} T^2 = -nm/(m+n)$ when we take $\alpha = \mathrm{i}\sqrt{mn/2(m+n)}$ in equation (16).

The immersion formulation also provides dualities between symmetries of the two models. For example, the discrete parity symmetry, $z \to -z$, exists both in the models. Similarly the transformation invariance of the linear equation in equation (9) under $\Psi(z,\bar{z},\lambda = 0) \to \Psi(z,\bar{z},\lambda = 0)\tilde{\Psi}$ ($\tilde{\Psi}$ is an arbitrary constant matrix) manifests itself as a rotational invariance of the immersion formula,

$$Q \to \tilde{\Psi}^{-1} Q \tilde{\Psi}. \tag{23}$$

4. Spin motions

In the following, we compute spin configurations of the generalized HF model using corresponding solutions of the Hermitian symmetric space NS model.

4.1. Trivial vacuum solution

The simplest solution of the HF model arises when we take the vacuum solution $E = 0$ of the NS models in equation (9), which gives

$$\Psi_f(\lambda) = \begin{pmatrix} \exp\left(-\frac{\mathrm{i}}{2}\lambda z + \frac{\mathrm{i}}{2}\lambda^2\bar{z}\right) I_{m\times m} & 0 \\ 0 & \exp\left(\frac{\mathrm{i}}{2}\lambda z - \frac{\mathrm{i}}{2}\lambda^2\bar{z}\right) I_{n\times n} \end{pmatrix}. \tag{24}$$

By inserting $\Psi_f(\lambda = 0)$ in equation (15), we obtain $Q = T$. Thus, using equation (16), we can get $Q = \frac{\mathrm{i}}{3}\left(\frac{3}{2}\lambda_3 + \frac{\sqrt{3}}{2}\lambda_8\right)$, i.e., $S_3 = \sqrt{3}/2$, $S_8 = 1/2$ for AIII type-I ($m = 1, n = 2$) theory. Similarly $S_1 = S_2 = 0$, $S_3 = -i$ for AIII type-II ($m = n = 1$) theory. Note that the normalization conditions for the spin variable of the two models are $\sum S_i^2 = 1$ and $S_1^2 + S_2^2 - S_3^2 = 1$, respectably. The rotational symmetry in equation (23) permits us to obtain other rotated spin configurations.

4.2. 1-soliton solution

A well-known vortex configuration in fluid mechanics is the solitary kink wave that propagates along a straight vortex filament line. In this section, we will describe a similar configuration of spin dynamics. The solitonic configuration of the HF model in N-dimension is obtained by dualizing the 1-soliton solution of the NS equation.

[1] In fact the term $-2\lambda\partial Q$ can be removed by a Galilean transformation with velocity v, i.e. $z' = z - v\bar{z}$, $\bar{z}' = \bar{z}$, $\lambda' = \lambda - \frac{1}{4}v$, which will set the parameter λ to zero, see [7] for detail. This paper also explains the R-transformation that relates the linear eigenvalue problem with an eigenvalue λ of the NS equation to that of the HF model with a new eigenvalue $\tilde{\lambda}$.

Applying the DT in equation (11) on $\Psi = \Psi_f(\lambda = 0)$ in equation (24), we obtain (with the DT parameter $\lambda_1 = -\mathrm{i}\eta\mathrm{e}^{-\mathrm{i}\theta}$)

$$\Psi[N](\lambda = 0) = 2\mathrm{i}\eta\cos\theta\frac{\Phi\Phi^\dagger\Sigma}{\Phi^\dagger\Sigma\Phi} - \mathrm{i}\eta\mathrm{e}^{\mathrm{i}\theta} \tag{25}$$

where

$$\Phi = \Psi_f(\lambda = \lambda_1 = -\mathrm{i}\eta\mathrm{e}^{-\mathrm{i}\theta})\Psi_0 \tag{26}$$

with arbitrary constant column matrix $\Psi_0 = \begin{pmatrix} C \\ D \end{pmatrix}$. ($C$ and D are m and n column matrices, respectively.) A straightforward calculation using equation (15) gives

$$\begin{aligned} Q = T &- \mathrm{i}\cos\theta\mathrm{e}^{-\mathrm{i}\theta}W\begin{pmatrix} CC^\dagger\mathrm{e}^{-X} & CD^\dagger\sigma\mathrm{e}^{\mathrm{i}Y} \\ -DC^\dagger\mathrm{e}^{-\mathrm{i}Y} & -DD^\dagger\sigma\mathrm{e}^{X} \end{pmatrix} \\ &-\mathrm{i}\cos\theta\mathrm{e}^{\mathrm{i}\theta}W\begin{pmatrix} CC^\dagger\mathrm{e}^{-X} & -CD^\dagger\sigma\mathrm{e}^{\mathrm{i}Y} \\ DC^\dagger\mathrm{e}^{-\mathrm{i}Y} & -DD^\dagger\sigma\mathrm{e}^{X} \end{pmatrix} \\ &+2\mathrm{i}\cos^2\theta W^2(C^\dagger C\mathrm{e}^{-X} - D^\dagger\sigma D\mathrm{e}^{X})\begin{pmatrix} CC^\dagger\mathrm{e}^{-X} & CD^\dagger\sigma\mathrm{e}^{\mathrm{i}Y} \\ DC^\dagger\mathrm{e}^{-\mathrm{i}Y} & DD^\dagger\sigma\mathrm{e}^{X} \end{pmatrix} \end{aligned} \tag{27}$$

where $W^{-1} = C^\dagger C\mathrm{e}^{-X} + D^\dagger\sigma D\mathrm{e}^{X}$, $X = \eta(\cos\theta)z + \eta^2(\sin 2\theta)\bar{z}$ and $Y = \eta(\sin\theta)z - \eta^2(\cos 2\theta)\bar{z}$. This is the most general expression of the 1-soliton solution of the HF model. To explore the result more explicitly, we proceed with the following examples.

Example 1. AIII type-I ($m = 1, n = 2$).
Taking

$$C = \sqrt{\frac{\alpha\mathrm{e}^{-b}}{2}} \qquad D = \sqrt{\alpha\mathrm{e}^{b-\mathrm{i}c}/2}\begin{pmatrix} \cos\phi \\ \sin\phi\mathrm{e}^{-\mathrm{i}\delta/2} \end{pmatrix}. \tag{28}$$

in equation (27), we obtain

$$\begin{aligned} Q = T &- \frac{\sqrt{3}}{2}\mathrm{i}\cos^2\theta\mathrm{sech}^2(X+b)\sin^2\phi\lambda_8 - \frac{1+\cos^2\phi}{2}\mathrm{i}\cos^2\theta\mathrm{sech}^2(X+b)\lambda_3 \\ &+\mathrm{i}\cos\phi\sin\phi\cos^2\theta\mathrm{sech}^2(X+b)(\cos\delta\lambda_6 - \sin\delta\lambda_7) \\ &-\cos\theta\sin\theta\mathrm{sech}(X+b)\cos\phi\{\mathrm{i}[\cos(Y+c) \\ &-\cot\theta\tanh(X+b)\sin(Y+c)]\lambda_2 \\ &+\mathrm{i}[\cos(Y+c)\cot\theta\tanh(X+b) + \sin(Y+c)]\lambda_1\}. \end{aligned} \tag{29}$$

The real spin variables S_i are obtained using equation (16),

$$\begin{aligned} &S_1 + \mathrm{i}S_2 = -\sqrt{3}\cos\theta\sin\theta\cos\phi\mathrm{sech}(X+b)\{\cot\theta\tanh(X+b) + \mathrm{i}\}\mathrm{e}^{-\mathrm{i}(Y+c)} \\ &S_3 = -\frac{\sqrt{3}}{2}\{(1+\cos^2\phi)\mathrm{sech}^2(X+b) - 1\} \\ &S_4 + \mathrm{i}S_5 = -\sqrt{3}\cos\theta\sin\theta\sin\phi\mathrm{sech}(X+b)\{\cot\theta\tanh(X+b) + \mathrm{i}\}\mathrm{e}^{-\mathrm{i}(Y+c+\delta)} \\ &S_6 + \mathrm{i}S_7 = \sqrt{3}\cos^2\theta\cos\phi\sin\phi\mathrm{sech}^2(X+b)\mathrm{e}^{-\mathrm{i}\delta} \\ &S_8 = -\tfrac{1}{2}\{3\cos^2\theta\sin^2\phi\mathrm{sech}^2(X+b) - 1\}. \end{aligned} \tag{30}$$

It can be explicitly checked that they satisfy the spin equation,

$$\bar{\partial}S_k = \sqrt{\frac{4}{3}}\bar{f}_{ijk}S_i\partial^2 S_j \qquad i, j, k = 1, 8 \tag{31}$$

where $\bar{f}_{ijk}$ are the $SU(3)$ structure constants defined as $[\lambda_i,\ \lambda_j] = 2\mathrm{i}\bar{f}_{ijk}\lambda_k$. Note that this is the HF equation associated with the physically interesting Manakov model. The HF equation for the generalized Manakov model when $k = n$ in equation (6) is obtained from equation (31) using the structure constants of $SU(n+1)$ and the nomalization factor $\sqrt{2n/(n+1)}$.

Example 2. AIII type-II $m = n = 1$.
Taking $C = \sqrt{\frac{\alpha}{2}}\mathrm{e}^{-b/2}$, $D = \sqrt{\frac{\alpha}{2}}\mathrm{e}^{b/2-\mathrm{i}c}$ in equation (27), we can obtain pure imaginary-valued spin solutions S_i;

$$
\begin{aligned}
S_1 + \mathrm{i}S_2 &= 2\mathrm{i}\cos\theta\,\sin\theta\,\mathrm{csch}(X+b)\{1 - \mathrm{i}\cot\theta\,\coth(X+b)\}\mathrm{e}^{-\mathrm{i}(Y+c)} \\
S_3 &= -\mathrm{i}\big(1 + 2\cos^2\theta\,\mathrm{csch}^2(X+b)\big).
\end{aligned}
\tag{32}
$$

They satisfy the spin equation in a three-dimensional space of non-definite metric,

$$
\partial S_1 = \mathrm{i}S_2\partial^2 S_3 - \mathrm{i}S_3\partial S_2 \qquad \partial S_2 = \mathrm{i}S_3\partial^2 S_1 - \mathrm{i}S_1\partial S_3 \qquad \partial S_3 = -\mathrm{i}S_1\partial^2 S_2 + \mathrm{i}S_2\partial S_1. \tag{33}
$$

Note that a simple transformation $S_i \to -\mathrm{i}S_i$ makes the spin solutions real-valued, which satisfy the spin equation in (33) without i factors.

4.3. Plane wave solution

In this section, we will explain a spin configuration corresponding to the plane wave of the NS model. The vortex configuration known as the Kelvin wave in fluid dynamics has a similar structure. The plane wave solutions of the NS model are obtained by solving eigenvalue problems which are specific to each Hermitian symmetric space. We proceed with explicit examples.

Example 1. AIII type-I ($m = 1, n = 2$).
The plane wave solution for this model is

$$
M^{cw} = (a_1\mathrm{e}^{\mathrm{i}b_1 z + \mathrm{i}c_1\bar{z}}, a_2\mathrm{e}^{\mathrm{i}b_2 z + \mathrm{i}c_2\bar{z}}) \tag{34}
$$

where $c_i = b_i^2 - 2(|a_1|^2 + |a_2|^2)$, $i = 1, 2$ and $a_i(b_i)$ are arbitrary complex (real) numbers. To obtain a spin configuration in $\boldsymbol{su}(\mathbf{3})$-algebra space, we need a solution of equation (10) at $\lambda = 0$. For this, we introduce new variables φ_i and rewrite the linear problem in equation (10) using these new variables;

$$
\begin{aligned}
\Psi_1 &= \exp\big(-\mathrm{i}(|a_1|^2 + |a_2|^2)\bar{z}\big)\varphi_1 \equiv \mathrm{e}^{-\Delta_1}\varphi_1, \\
\Psi_2 &= \begin{pmatrix} \exp\big(-\mathrm{i}b_1 z + \mathrm{i}(|a_1|^2 + |a_2|^2 - b_1^2)\bar{z}\big)\varphi_2 \\ \exp\big(-\mathrm{i}b_2 z + \mathrm{i}(|a_1|^2 + |a_2|^2 - b_2^2)\bar{z}\big)\varphi_3 \end{pmatrix} \equiv \begin{pmatrix} \mathrm{e}^{-\Delta_2}\varphi_2 \\ \mathrm{e}^{-\Delta_3}\varphi_3 \end{pmatrix}
\end{aligned}
\tag{35}
$$

such that

$$
\partial\varphi + U\varphi = 0 \qquad \bar{\partial}\varphi + V\varphi = 0 \tag{36}
$$

where

$$
U = \begin{pmatrix} 0 & a_1 & a_2 \\ -a_1^* & -\mathrm{i}b_1 & 0 \\ -a_2^* & 0 & -\mathrm{i}b_2 \end{pmatrix} \qquad V = \begin{pmatrix} 0 & a_1 b_1 & a_2 b_2 \\ -a_1^* b_1 & \mathrm{i}|a_2|^2 - \mathrm{i}b_1^2 & -\mathrm{i}a_1^* a_2 \\ -a_2^* b_2 & -\mathrm{i}a_2^* a_1 & \mathrm{i}|a_1|^2 - \mathrm{i}b_2^2 \end{pmatrix}. \tag{37}
$$

These two matrices commute, i.e., $[U,\ V] = 0$. Thus, by solving the eigenvalue problem, we may diagonalize U and V simultaneously in terms of an 3×3 matrix D such that

$$
\begin{aligned}
D^{-1}UD &= \mathrm{diag}(p_1, p_2, p_3) \\
D^{-1}VD &= \mathrm{diag}(q_1, q_2, q_3).
\end{aligned}
\tag{38}
$$

Now the solution of the linear equation becomes

$$
\Psi = \mathrm{diag}(\mathrm{e}^{-\Delta_1}, \mathrm{e}^{-\Delta_2}, \mathrm{e}^{-\Delta_3})\ D\ \mathrm{diag}(\mathrm{e}^{-p_1 z - q_1\bar{z}}, \mathrm{e}^{-p_2 z - q_2\bar{z}}, \mathrm{e}^{-p_3 z - q_3\bar{z}}). \tag{39}
$$

The explicit expressions of p_i, q_i, and D matrix are quite messy as they contain a solution of a cubic equation. Here we present results that have simple expression but still have the essential feature of spin motions in $\boldsymbol{su}(\mathbf{3})$ space. Taking $b_1 = |a_1|^2, b_2 = -|a_2|^2$ and using $Q = \Psi^{-1}T\Psi = \mathrm{i}S_i\lambda_i/\sqrt{3}$, we can obtain

$$
\begin{aligned}
S_1 + \mathrm{i}S_2 &= \sqrt{\frac{3(p+q)}{(4+p+q)(pq+p+q)}}\exp\left(\mathrm{i}Rz + \mathrm{i}(p-q)R\bar{z}\right) \\
S_3 &= -\frac{\sqrt{3}(p-q)}{2M_t(pq+p+q)} \\
S_4 + \mathrm{i}S_5 &= \sqrt{\frac{6pq}{(pq+p+q)(4+p+q+M_t p - M_t q)}} \\
&\quad \times \exp\left(\mathrm{i}R_+ z + \mathrm{i}((p-q)R_+ + pq + p + q)\bar{z}\right) \\
S_6 + \mathrm{i}S_7 &= \sqrt{\frac{6pq}{(pq+p+q)(4+p+q-M_t p + M_t q)}} \\
&\quad \times \exp\left(\mathrm{i}R_- z + \mathrm{i}((p-q)R_- + pq + p + q)\bar{z}\right) \\
S_8 &= -\frac{2pq - p - q}{2pq + 2p + 2q}
\end{aligned}
\tag{40}
$$

where $p = |a_1|^2, q = |a_2|^2$, $R = \sqrt{(p+q)^2 + 4(p+q)}$, $M_t = R/(p+q)$, $R_\pm = \frac{p-q\pm R}{2}$.

Example 2. AIII type-II ($m = n = 1$).
The plane wave solution in this case is

$$M^{cw} = a\mathrm{e}^{\mathrm{i}bz + \mathrm{i}c\bar{z}} \tag{41}$$

where $c = b^2 + 2|a|^2$, with a complex parameter a and a real parameter b. Taking a similar procedure as in (example 1), we arrive at the following expression for the solution of the linear equation;

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\Delta/2}, \mathrm{e}^{-\mathrm{i}\Delta/2}) \begin{pmatrix} p & -a \\ -a^* & -p \end{pmatrix} \mathrm{diag}(\mathrm{e}^{-\mathrm{i}BX}, \mathrm{e}^{\mathrm{i}BX}) \tag{42}$$

where $B = \sqrt{b^2/4 - |a|^2}$, $\Delta = bz + c\bar{z}$, $p = -\frac{\mathrm{i}}{2}b - \mathrm{i}B$ and $X = z + b\bar{z}$. Now, using $Q = \Psi^{-1}T\Psi = \mathrm{i}S_iT_i/2$, we obtain a pure imaginary-valued spin configuration,

$$
\begin{aligned}
S_1 + \mathrm{i}S_2 &= \frac{-2}{p^2 + |a|^2}pa^*\exp(-2\mathrm{i}BX) \\
S_3 &= \frac{-\mathrm{i}}{p^2 + |a|^2}(p^2 - |a|^2).
\end{aligned}
\tag{43}
$$

Note that they indeed satisfy $S_1^2 + S_2^2 - S_3^2 = 1$.

5. Discussion

In this paper, we have explained the ‘soliton surface’ theoretic formulation of the HF model in N-dimension that is dual to the Hermitian symmetric space NS model. It gives an explicit method to calculate spin configurations of the N-dimensional HF model, like the vacuum, the plane wave and the 1-soliton.

Our formalism can be easily adapted to describe the duality between the vortex dynamics of fluid mechanics in N-dimension and the Hermitian symmetric space NS equations. In fact, the Da Rios–Betchov equation, $\dot{X} = X' \times X''$, which is the dynamical equation for a vortex $X(\tau, \sigma)$, is related to the spin equation, $\dot{S} = S \times S''$, via $S = X'$. Thus we can find vortex configurations of fluid mechanics in N-dimension by integrating spin configurations with respect to σ. This generalizes the Hasimoto's work [10] to N-dimension without relying on the space-curve formalism.

The stability analysis of the obtained spin (or vortex) configurations was partially conducted using the perturbation method in three dimension [16]. Our duality formalism could be important for the stability analysis of spin configurations, as there exists a method for the analysis of long-time behavior of instabilities using the (quasi-)periodic solution of the NS system [17].

Finally we note that our Sym–Tafel type formula can also be applied for the analysis of the $SU(3)$ spin chain equation and the three component NS-like equation considered in [18]. This three component equation can be shown to be a generalized NS equation associated with a reductive homogeneous space [8]. As the symmetric spaces are subspaces of the homogeneous spaces, the Hermitain symmetric space NS equations considered in this paper can be obtained from a reduction of the homogeneous space NS equations.

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